10.

Pf: If $u_n \le v_n$, then

$$v_n \ge v_{n+1} = \frac{u_n+v_n}{2} \overset{\swarrow \text{ fundamental inequality.}}{\ge} \sqrt{u_n v_n} = u_{n+1} \ge u_n.$$

Since $u_1 = \sqrt{u_0 v_0} \le \frac{u_0+v_0}{2} = v_0$, hence we have

$$v_1 \ge v_2 \ge \cdots \ge v_n \ge u_n \ge \cdots \ge u_2 \ge u_1, \ \forall n \in \mathbb{N}_+$$

$\Rightarrow$ $\{u_n\}$ is increasing and $u_n \le v_0$, $\forall n$. $\Rightarrow$ $\lim\limits_{n\to\infty} u_n$ exists.

$\{v_n\}$ is decreasing and $v_n \ge u_0$, $\forall n$. $\lim\limits_{n\to\infty} v_n$ exists.

$\Rightarrow$ $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{u_n+v_n}{2} = \frac{\lim\limits_{n\to\infty} u_n + \lim\limits_{n\to\infty} v_n}{2} \Rightarrow \lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} v_n.$

12.

Pf: $u_n = \left(\frac{1}{n} - \frac{1}{n+1}\right)\frac{1}{n+2} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) - \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{\frac{1}{2}}{n} - \frac{1}{n+1} + \frac{\frac{1}{2}}{n+2}$ i.e. $a = \frac{1}{2}, b = -1, c = \frac{1}{2}$.

$= \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

$$v_n = \sum_{k=1}^{n} u_k = \frac{1}{2}\left(\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) - \sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+2}\right)\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{n+1} - \frac{1}{2} + \frac{1}{n+2}\right) = \frac{1}{4} + \frac{1}{2}\left(-\frac{1}{n+1} + \frac{1}{n+2}\right)$$

$\Rightarrow \lim\limits_{n\to\infty} v_n = \frac{1}{4}$

14.

Pf: Let $\{a_k\}_{k=1}^{\infty}$ be a sequence which has no upper bound.

$\forall n \in \mathbb{N}$, $\exists k_n \in \mathbb{N}$ s.t. $a_{k_n} \ge n$, otherwise $a_k \le n$, $\forall k \in \mathbb{N}$, which is a contradiction

$\Rightarrow \{a_{k_n}\}_{n=1}^{\infty}$ is a subsequence which diverges to $+\infty$.

16

Pf: (1) $v_{n+1} - v_n$

$$= \frac{1}{(n+1)!} + \frac{1}{(n+1)(n+1)!} - \frac{1}{n \times n!}$$

$$= \frac{n(n+1) + n - (n+1)^2}{n(n+1)\cdot(n+1)!} = -\frac{1}{n(n+1)\cdot(n+1)!} \le 0$$

$$\Rightarrow \quad u_1 \le \cdots \le u_{n-1} \le u_n \le v_n \le v_{n-1} \le \cdots \le v_1, \ \forall n$$

$$\Rightarrow \quad \lim_{n\to\infty} u_n, \ \lim_{n\to\infty} v_n \text{ exist and } \lim_{n\to\infty} u_n - \lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n\cdot n!} = 0$$

(2) Let $l = \lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$. Suppose that $l = \frac{p}{q} \in \mathbb{Q}$ s.t. p, q are coprime integers.

$$\Rightarrow \quad u_{q-1} + \frac{1}{q!} < \frac{p}{q} < u_{q-1} + \frac{1}{q!} + \frac{1}{q\cdot q!}$$

$$\Rightarrow \quad \frac{q\cdot q!\cdot u_{q-1} + q}{q\cdot q!} < \frac{p\cdot q!}{q\cdot q!} < \frac{q\cdot q!\cdot u_{q-1} + (q+1)}{q\cdot q!} \Rightarrow p\cdot q! \in (q\cdot q!\, u_{q-1} + q, \ q\cdot q!\cdot u_{q-1} + q + 1)$$

Since $q\cdot q!\cdot u_{q-1}$ is a positive integer by the def of u_{q-1},

$(q\cdot q!\, u_{q-1} + q, \ q\cdot q!\cdot u_{q-1} + q - 1)$ contains no integer ! $\square$.

(3) $$\sum_{k=0}^{n} \frac{ak+b}{k!} = a\sum_{k=1}^{n} \frac{1}{(k-1)!} + b\sum_{k=0}^{n} \frac{1}{k!}$$
$$= a\sum_{k=0}^{n-1} \frac{1}{k!} + b\sum_{k=0}^{n} \frac{1}{k!} = a u_{n-1} + b u_n \to (a+b)l \text{ as } n\to\infty$$

18.

Pf: Since $|\sin \frac{1}{n^2}| < \frac{1}{n^2}$, we have

$$|u_n| = (5|\sin \tfrac{1}{n^2}| + \tfrac{1}{5}|\cos n|)^n$$
$$< (\tfrac{5}{n^2} + \tfrac{1}{5})^n$$
$$\overset{n>5}{<} (\tfrac{2}{5})^n$$

$$\Rightarrow \lim_{n\to\infty} u_n = 0$$

20.

Pf: $\forall \varepsilon > 0$, $\exists\, p$ s.t. $\alpha_p < \frac{\varepsilon}{2}$, take $N = [\frac{2p}{\varepsilon} - 1] + 1$ i.e. $\frac{p}{N+1} \le \frac{\varepsilon}{2}$, then

$$|u_n| \le \alpha_p + \frac{p}{n+1}$$
$$< \frac{\varepsilon}{2} + \frac{p}{N+1} \le \varepsilon, \ \forall n \ge N$$

$\square$.

22.

Pf: Let $l = \lim_{n\to\infty} u_n$.

$$u_n - \frac{n^2+n}{2n^2}\cdot l = \frac{(u_1-l)+2(u_2-l)+\cdots n(u_n-l)}{n^2}$$

$\forall \varepsilon>0$, $\exists N_1 \in \mathbb{N}$ s.t. $\forall n\geq N_1$ $|u_n-l|\leq \frac{\varepsilon}{2}$, then

$$\left|\frac{N_1(u_{N_1}-l)+\cdots+n(u_n-l)}{n^2}\right|$$

$$\leq \frac{N_1+\cdots+n}{n^2}\cdot\frac{\varepsilon}{2} \leq \frac{\frac{n^2+n}{2}}{n^2}\cdot\frac{\varepsilon}{2} = \left(\frac{1}{2}+\frac{1}{2n}\right)\cdot\frac{\varepsilon}{2}\leq\frac{\varepsilon}{2}, \quad \forall n\geq N_1$$

Since $(u_1-l)+2(u_2-l)+\cdots+(N_1-1)(u_{N_1-1}-l)$ is finite, $\exists N_2$ s.t. $\forall n \geq N_2$

$$\frac{|(u_1-l)+2(u_2-l)+\cdots+(N_1-1)(u_{N_1-1}-l)|}{n^2}\leq\frac{\varepsilon}{2}$$

$\Rightarrow \forall n\geq \max\{N_1,N_2\}$, $\left|u_n-\frac{n^2+n}{2n^2}\cdot l\right|\leq\varepsilon$

$\Rightarrow \lim_{n\to\infty}u_n = \lim_{n\to\infty}\frac{n^2+n}{2n^2}\cdot l = \frac{l}{2}$ □

24.

Pf: (1). Since u_n diverges to $+\infty$, $\exists n_1>n_0$ s.t. $u_{n_1}>x$, then

$$u_{n_0}\leq x<u_{n_1}.$$

$A=\{n>n_0 \mid u_n>x\}\neq\phi$ has infimum n_2

$\Rightarrow n_2-1\notin A \Rightarrow u_{n_2-1}\leq x<u_{n_2} \xRightarrow{|u_{n_2-1}-u_{n_2}|<\varepsilon} |u_{n_2}-x|<\varepsilon$

(2). $\forall x\in\mathbb{R}$, $\exists m\in\mathbb{N}$ s.t. $v_m+x\geq u_{n_0}$.

By Ex(1), $\exists p\in\mathbb{N}$ s.t. $|u_p-(v_m+x)|\leq\varepsilon$ □.

26.

Pf. (1) $u_n=\sqrt{n+2}-\sqrt{n+1}=\frac{1}{\sqrt{n+2}+\sqrt{n+1}}\sim\frac{1}{2\sqrt{n}}$

(2) $v_n=e^{\frac{1}{n}}-e^{\frac{1}{n+1}}\sim\left(1+\frac{1}{n}\right)-\left(1+\frac{1}{n+1}\right)\sim\frac{1}{n^2}$

(3) $w_n=\sqrt{1+\frac{1}{\ln(n+1)}}-1=\frac{\frac{1}{\ln(n+1)}}{\cdots}\sim\frac{1}{2\cdot\ln(n+1)}$

$1+\sqrt{1+\frac{1}{\ln(n+1)}}$ $\quad$ $2\ln(n+1)$

28.

解: (1) $\sqrt{k+1}-\sqrt{k} = \dfrac{k+1-k}{\sqrt{k+1}+\sqrt{k}} = \dfrac{1}{\sqrt{k+1}+\sqrt{k}}$

$\Rightarrow \dfrac{1}{2\sqrt{k+1}} \le \sqrt{k+1}-\sqrt{k} \le \dfrac{1}{2\sqrt{k}}$

(2).

By (1), $\sqrt{k+1}-\sqrt{k} \le \dfrac{1}{2\sqrt{k}} \le \sqrt{k}-\sqrt{k-1}$,

$\Rightarrow \quad 2\sqrt{n+1}-2 \le u_n \le 2\sqrt{n}$

$\Rightarrow \sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}} \le \dfrac{u_n}{2\sqrt{n}} \le 1$

$\Rightarrow \quad \lim\limits_{n\to\infty} \dfrac{u_n}{2\sqrt{n}} = 1$

$\Rightarrow u_n \sim 2\sqrt{n}$ $\square$

30.

解: $1 \le \dfrac{\sum\limits_{k=1}^{n} k!}{n!} = 1+\dfrac{1}{n}+\sum\limits_{k=1}^{n-2}\dfrac{k!}{n!}$

$\le 1+\dfrac{1}{n}+\sum\limits_{k=1}^{n-2}\dfrac{(n-2)!}{n!}$

$= 1+\dfrac{1}{n}+\dfrac{n-2}{n(n-1)}$

$\le 1+\dfrac{1}{n}+\dfrac{1}{n} = 1+\dfrac{2}{n}$

$\Rightarrow \lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=1}^{n}k!}{n!} = 1 \quad \Rightarrow \sum\limits_{k=1}^{n} k! \sim n!$